MULTI–SOLITONS AND LARGE TIME DYNAMICS OF THE SUBCRITICAL GENERALIZED KDV EQUATIONS

YVAN MARTEL

Abstract. We consider the generalized Korteweg–de Vries equations (gKdV)

$$\partial_t u + \partial_x(\partial_x^2 u + u^p) = 0 \quad t, x \in \mathbb{R},$$

in the subcritical case $1 < p < 5$. We review recent results on the large time behavior of the solutions of the subcritical gKdV equations which are in an H^1 neighborhood of one soliton or of the sum of several solitons. Such results are motivated by the remarkable properties of the N-soliton solutions of the KdV and mKdV equations (i.e. $p = 2$ and 3 for which the equation is completely integrable). Similar questions for a few other nonlinear dispersive equations with solitary waves are discussed.

1. Introduction

In these notes, we review recent results concerning the long time behavior of solutions of the subcritical generalized Korteweg–de Vries equations (gKdV equations)

$$\partial_t u + \partial_x(\partial_x^2 u + u^p) = 0 \quad t, x \in \mathbb{R}, \tag{1.1}$$

($1 < p < 5$) which are close to one soliton or to the sum of several solitons. These results concern mainly the stability and asymptotic stability of the family of solitons and the existence, uniqueness and stability of multi-soliton type solutions. We also discuss similar questions for some other nonlinear dispersive equations, mainly for some nonlinear Schrödinger equations.

Questions related to solitons and multi-solitons of the generalized KdV equations being motivated by the remarkable properties of the original KdV equation ($p = 2$), we start by recalling some of these properties.

1.1. **Integrability and multi-solitons.**

It is well-known that the Korteweg-de Vries equation

$$\partial_t u + \partial_x(\partial_x^2 u + u^2) = 0 \quad t, x \in \mathbb{R}, \tag{1.2}$$

is completely integrable, which means that it has remarkable features for a nonlinear partial differential equation (see e.g. Lax [33], Lamb [31], Miura [54] and Schuur [63]). We recall in this section some classical results concerning multi-solitons for the KdV equation. Many of these properties were obtained using the Inverse Scattering Transform.

First, there are formally an *infinite number of conservation laws* for a solution $u(t)$ of the KdV equation

$$\int u(t,x)dx, \quad \int u^2(t,x)dx, \quad \int \left([\partial_x u(t,x)]^2 - \tfrac{2}{3}u^3(t,x)\right) dx,$$

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$$\int \left([\partial_x^2 u(t,x)]^2 - \tfrac{10}{3}[\partial_x u(t,x)]^2 u(t,x) + \tfrac{5}{9}u^4(t,x)\right) dx, \quad \text{etc.}$$

Higher order conserved quantities require solutions in H^s, for $s \geq 3$ in order to be defined. Complete integrability is also related to the discovery for the KdV equation of a large family of explicit *multi-soliton* solutions. To get started, note that the function $U_1(t,x)$ defined by

$$U_1(t,x) = \tfrac{3}{2}\cosh^{-2}\left(\tfrac{x-t}{2}\right) = 6\tfrac{\partial^2}{\partial x^2}\log\left(1+e^{x-t}\right)$$

is a travelling wave solution of (1.2). All other travelling waves are obtained from U_1 by scaling and translation. Indeed, denoting

$$Q(x) = \tfrac{3}{2}\cosh^{-2}\left(\tfrac{x}{2}\right), \quad Q_c(x) = cQ(\sqrt{c}x) \quad \text{and} \quad U_c(t,x) = Q_c(x-ct),$$

the travelling waves solutions of (1.2) are exactly $U_c(t, x-x_0)$, for all $c > 0$, $x_0 \in \mathbb{R}$. Note that

$$\int Q_c^2 = c^{3/2}\int Q^2, \tag{1.3}$$

which means that the speed of the travelling wave is related to its size in L^2.

The following function $U_{1,c}$ which is a solution of (1.2), is a typical 2-soliton solution of (1.2) ($0 < c < 1$):

$$U_{1,c}(t,x) = 6\frac{\partial^2}{\partial x^2}\log\left(1 + e^{x-t} + e^{\sqrt{c}(x-ct)} + \alpha e^{x-t}e^{\sqrt{c}(x-ct)}\right),$$

with $\alpha = \left(\frac{1-\sqrt{c}}{1+\sqrt{c}}\right)^2$. By a classical analysis (Wadati and Toda [66]), we have

- As $t \to +\infty$:
  $$\|U_{1,c}(t,x) - Q_c(x-ct) - Q(x-t-\delta)\|_{H^1(\mathbb{R})} \to 0,$$
  where $\delta = 2\log\left(\frac{1+\sqrt{c}}{1-\sqrt{c}}\right) > 0$.
- As $t \to -\infty$:
  $$\|U_{1,c}(t,x) - Q_c(x-ct-\delta') - Q(x-t)\|_{H^1(\mathbb{R})} \to 0,$$
  where $\delta' = \frac{\delta}{\sqrt{c}} > 0$.

This solution contains two solitary waves Q and Q_c whose sizes are unchanged by interaction. The terminology *solitons* was introduced by Zabusky and Kruskal [70] to refer to travelling waves having such a stability property through interaction. Note that the trajectories of the solitary waves are shifted, as can be seen on the limit behavior as $t \to +\infty$ and $t \to -\infty$ ($\delta, \delta' > 0$).

More general but similar expressions for multi-soliton solutions can be written for any number of solitons. Indeed, for any given parameters $0 < c_1 < \ldots < c_N$, $\delta_1, \ldots, \delta_N \in \mathbb{R}$, there exists an explicit multi-soliton solution $U(t,x)$ that satisfies as $t \to +\infty$:

$$\left\|U(t,x) - \sum_{j=1}^{N} Q_{c_j}(x - c_j t - \delta_j)\right\|_{H^1} \to 0.$$

Moreover, such solution also satisfies as $t \to -\infty$:

$$\left\|U(t,x) - \sum_{j=1}^{N} Q_{c_j}(x - c_j t - \delta_j')\right\|_{H^1} \to 0,$$

for some δ_j' such that $\delta_j - \delta_j'$ depends on all the (c_k).

These special solutions are fundamental in studying the properties of general solutions of the KdV equation because of the so-called *decomposition property.* Recall that Eckhaus and Schuur [17] state rigorously this property for any initial data such that the Inverse Scattering Transforms applies. More precisely, they proved the following result.

Theorem (Cohen [10]; Eckhaus and Schuur [17], [63])**.** *Suppose that* $u(0,x)$ *satisfies*

$$u(0) \in C^4(\mathbb{R}), \quad k=0,\ldots,4,\ \forall x \in \mathbb{R}, \quad \left|\frac{\partial^k u(0,x)}{\partial x^k}\right| \leq \frac{C}{|x|^{10}}.$$

Then, for $x>0$*, as* $t \to +\infty$*,*

$$u(t,x) - \sum_{j=1}^{N} Q_{c_j}(x - x_j - c_j t) \to 0.$$

This result means that the asymptotic behavior of any solution (sufficiently regular and decaying) is governed by a finite number of solitons. The number N of solitons emerging is related to a spectral property of $u(0)$. We refer to the original paper and book for a precise convergence result.

Finally, we recall another rigorous result concerning N-soliton solutions of the KdV equation. The N-soliton solutions of (1.2) are stable in H^N, which means that if a solution $u(t) \in H^N(\mathbb{R})$ of the KdV equation is such that for some time $t_0 \in \mathbb{R}$, $u(t_0)$ is close to such an N-soliton solution in $H^N(\mathbb{R})$ then for all $t \in \mathbb{R}$, $u(t)$ is close in $H^N(\mathbb{R})$ to an N-soliton solution with same speeds (but possibly different positions). This result is to due to Maddocks and Sachs [36] and we refer to this paper for a more precise statement. The exponent of the Sobolev norm has to be larger or equal to the number of solitons since the proof of the stability result uses the first $N+1$ conserved quantities, and thus does not hold in the energy space H^1 for $N \geq 2$.

There are other nonlinear partial differential equations coming from Physics that are completely integrable, see e.g. the book of Ablowitz and Segur [1]. Let us mention a few of them that have mathematical similarities with the KdV equation.

– First, of course, the modified KdV equation, i.e. equation (2.1) with $p=3$ is the closest model to the KdV equation. It is also a completely integrable equation and also admits explicit multi-soliton solution (see Ohmiya [57] and the book of Schuur [63]). Note that the mKdV equation also has *breather solutions*, i.e. another large family of explicit solutions that are localized. These solutions may also appear asymptotically in the behavior of general solutions, but it seems that their nonlinear stability was not studied so far.

– Second, consider the one dimensional cubic Schrödinger equation,

$$\mathrm{i}\partial_t u + \partial_x^2 u + |u|^2 u = 0 \quad t, x \in \mathbb{R}. \tag{1.4}$$

Recall that for a solitary wave of (1.4), the speed and the scaling are not related, as it is the case for the KdV equation, thus the two parameters can be chosen independently and the family of solitary waves is larger. See sections 2.3 and 2.6 for a description of the family of travelling waves. Zakharov and Shabat [71] have proved the existence of multi-soliton solutions for this equation. From [71], one knows that the family of such multi-solitons is richer than for the KdV equation : first, simply as a consequence of having a larger set of solitary waves, and also because multi solitary waves containing parallel solitons exist. We will discuss in these notes some generalized versions of (1.4) (general nonlinearities and $x \in \mathbb{R}^d$, $d \geq 1$).

– The so called KP I and KP II equations are also completely integrable models:

$$u_t + \partial_x^3 u \pm 3\partial_x^{-1}\partial_y^2 u + 6u\partial_x u = 0 \quad (t,x) \in \mathbb{R} \times \mathbb{R}^2. \tag{1.5}$$

Recall that for the + sign in front of the third term, this is the KP II equation that does not admit solitary waves (see de Bouard and Saut [14] and also de Bouard and Martel [13]). The KP I equation does admit solitary waves (see de Bouard and Saut [14], [16]) and their stability has been studied, but to our knowledge, questions related to multi-soliton are open. We will not further discuss the KP equations in these notes.

1.2. **Questions for the generalized models.** The generalized KdV equation (1.1) is known to be integrable only for $p = 2$ or 3. For a different power in the nonlinearity (or any other perturbation of the model), the Inverse Scattering Transform does not apply and the properties related to multi-solitons are not known. In particular, explicit multi-soliton solutions cannot be expected for the non integrable models. A vast question is to what extend the remarkable properties of solitons and multi-solitons of the integrable equations persist for the generalized models. These notes are devoted to the presentation of some recent results concerning this question. Surprizingly, some of these results also give some more insight on the behavior of the solutions around multi-solitons *even in the integrable cases.*

The main conclusion of the results gathered in these notes is that the properties of the flow of the gKdV equations in *a neighborhood of one or several solitons*, and *asymptotically as* $t \to +\infty$ are quite similar to the ones of the KdV equation. Since the techniques developed in this situation do not use the very specific structure of the KdV equation, we believe that they have a wide scope of applicability. To support this belief, we present some extensions of these techniques to some other equations with solitary waves.

Since we consider only large time properties, and solutions containing several solitons with different speeds, the solitons are always decoupled in the results presented here. It seems that so far in the non-integrable cases, no information is available for the *interaction of several solitons* i.e. when the solitons come closer and begin to interact. In particular, it is not clear whether such interaction should preserve the shape of the solitons. By abuse of language, we may still call *solitons* the travelling waves of the gKdV equations.

Exactly the same questions can be asked for generalizations of the one dimensional cubic Schrödinger equation. We present in these notes the latest results in this direction.

1.3. **Outline of the paper.** In Section 2, we recall the generalized models to be studied and some of the general results on these models : local and global in time well-posedness results, existence of travelling wave solutions, stability or asymptotic stability results of these solutions.

In the next sections, we present recent results on the gKdV models which appeared in [42], [48], [38] and [46]. Results for nonlinear Schrödinger equations and for a nonlinear equations introduced by Peregrine [59] and Benjamin, Bona and Mahony [3] are also presented ([49], [47], [18], [19], [20] and [39]). All these results hold in the energy space, and *are not based on complete integrability* and the Inverse Scattering Transform. Note that in this paper, we consider only H^1 global and uniformly bounded solutions.

In Section 3, we recall a result of asymptotic stability of one soliton of the gKdV equation that holds in the energy space and we give a sketch of its proof.

Sections 4 and 5 deal with the case of multi-soliton solutions, i.e. solutions which are close to the sum of several solitons of different speeds for large time (i.e. when the solitons are all sufficiently decoupled). Section 4 is concerned with the existence of multi-soliton type solutions as $t \to +\infty$, and section 5 present some results concerning the stability and asymptotic stability of such solutions.

In Sections 3, 4 and 5, we also present similar results for two other dispersive equations, the nonlinear Schrödinger equation and a nonlinear equation introduced in [59] and [3] which is similar to the KdV equation.

These notes do not cover the critical case $p = 5$ in (2.1) and the blow-up phenomenon that was described in [41], [43], [51] and [44]. We refer the interested reader to these papers or to the review papers [45], Tzevtkov [65].

For recent progress in the blow up phenomenon for the critical Schrödinger equation in $\mathbb{R}^d$, we refer to Merle and Raphaël [52] and to references therein.

2. CLASSICAL RESULTS FOR THE GENERALIZED MODELS

2.1. Introduction of the generalized models.

In this section, we present the models to be studied in these notes.

• Generalized Korteweg-de Vries equations : As mentioned in the Introduction, we mainly focus on the generalized KdV equation with power nonlinearity :

$$\partial_t u + \partial_x(\partial_x^2 u + u^p) = 0, \quad t, x \in \mathbb{R}, \tag{2.1}$$

for $p = 2, 3, 4$. Formally, the conserved quantities for (2.1) are for all $p > 1$,

$$\int u(t) = \int u(0), \tag{2.2}$$

$$\int u^2(t) = \int u^2(0), \tag{2.3}$$

$$E(u(t)) = \frac{1}{2}\int (\partial_x u(t))^2 - \frac{1}{p+1}\int u^{p+1}(t) = E(u(0)). \tag{2.4}$$

Except for $p = 2$ and 3, there are no other conserved quantities. Recall the symmetries of the gKdV equation :

- Scaling invariance : if $u(t, x)$ is solution of (2.1) then $c_0^{\frac{1}{p-1}} u(c_0^{\frac{3}{2}} t, c_0^{\frac{1}{2}} x)$ is also solution, for any $c_0 > 0$,
- Translation invariance : if $u(t, x)$ is solution of (2.1) then $u(t - t_0, x - x_0)$ is also solution, for any t_0, $x_0 \in \mathbb{R}$.

We will also consider, more generally,

$$\partial_t u + \partial_x(\partial_x^2 u + g(u)) = 0, \quad t, x \in \mathbb{R}, \tag{2.5}$$

where g is of class C^1 and $g(0) = 0$.

• Nonlinear Schrödinger equations in $\mathbb{R}^d$: First, we consider the pure power nonlinear Schrödinger equations ($d \geq 1$):

$$\mathrm{i}\partial_t u + \Delta u + |u|^{p-1}u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \tag{2.6}$$

for

$$1 < p < \frac{d+2}{d-2}, \text{if } d \geq 3,$$

with conserved quantities

$$\int |u(t)|^2 = \int |u(0)|^2, \tag{2.7}$$

$$E(u(t)) = \frac{1}{2}\int |\nabla u(t)|^2 - \frac{1}{p+1}\int |u(t)|^{p+1} = E(u(0)), \tag{2.8}$$

$$\operatorname{Im}\int \nabla u(t)\overline{u}(t) = \operatorname{Im}\int \nabla u(0)\overline{u}(0). \tag{2.9}$$

Except for $d = 1$ and $p = 3$, there are no other conserved quantities for (2.6). We also list the symmetries of equation (2.6) :

- Space-time translation invariance: if $u(t,x)$ satisfies (2.6), then for any $t_0, x_0 \in \mathbb{R}$, $w(t,x) = u(t - t_0, x - x_0)$ also satisfies (2.6).
- Phase invariance: if $u(t,x)$ satisfies (2.6), then for any $\gamma_0 \in \mathbb{R}$, $w(t,x) = u(t,x)e^{\mathrm{i}\gamma_0}$ also satisfies (2.6).
- Galilean invariance: if $u(t,x)$ satisfies (2.6), then for any $v_0 \in \mathbb{R}$,

$$w(t,x) = u(t, x - v_0 t)e^{\mathrm{i}\frac{v_0}{2}(x - \frac{v_0}{2}t)} \tag{2.10}$$

also satisfies (2.6).

More generally, we also introduce

$$\mathrm{i}\partial_t u + \Delta u + f(|u|^2)u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \tag{2.11}$$

for f of class C^1, $f(0) = 0$ and satisfying

$$\text{for all } s \geq 1, \quad |f'(s^2)| < Cs^{p-2}, \quad \text{for some } 1 < p < \frac{d+2}{d-2}.$$

• Peregrine [59] and Benjamin, Bona and Mahony [3] have introduced another equation similar to the KdV equation. It writes as follows

$$(1 - \partial_x^2)\partial_t u + \partial_x(u^2) = 0, \quad t, x \in \mathbb{R}. \tag{2.12}$$

Note that in this case, the invariant quantities are

$$\int u^2(t) + (\partial_x u(t))^2 = \int u^2(0) + (\partial_x u(0))^2, \tag{2.13}$$

$$E(u(t)) = \frac{1}{2}\int u^2(t) + \frac{1}{p+1}\int u^{p+1}(t) = E(u(0)). \tag{2.14}$$

We could also consider in (2.12) a generalized nonlinearity of the form u^p or even $g(u)$ for some suitable g, however we restrict ourselves to the quadratic nonlinearity and we will refer to original papers for details on the generalized form. Note that (2.12) *is not completely integrable*, nor is any variant of it.

Note finally that the three models considered above are *time reversible*:
– If $u(t,x)$ is solution of (2.1) then so is $u(-t,-x)$;
– If $u(t,x)$ is solution of (2.6) then so is $\bar{u}(-t,x)$;
– If $u(t,x)$ is solution of (2.12) then so is $u(-t,-x)$.

2.2. **Cauchy problem in H^1, Kato - KPV (global, local).** For all the models and nonlinearities under consideration, the local Cauchy problem is well-posed in H^1, in a strong sense, i.e. with continuous dependence upon the initial data in H^1. Moreover, the persistence property holds, which means that if the initial data is in H^s, for some $s > 1$, then the solution is also in H^s. Moreover any H^1 solution considered is the limit in H^1 of more regular solutions obtained by regularization of the initial data. This allows us to justify the conservation of quantities defined in H^1 : (2.3)–(2.4) for the gKdV equation, (2.7)–(2.9) for the NLS equation and (2.13)–(2.14) for the P-BBM equation.

For the NLS equation, the resolution of the local Cauchy problem in H^1 is due to Ginibre and Velo [24], and for the gKdV equation to Kenig, Ponce and Vega [28] (see also references therein for previous works on the subject). For equation (2.12), the result is elementary due to regularizing properties of the operator $(1 - \partial_{x^2})^{-1}$, see [3]. We refer to these papers for a precise statement of the well-posedness results.

Under subcritical assumptions, i.e. $1 < p < 5$ for (2.1), or $1 < p < 1 + 4/d$ for (2.6), and similar assumptions for the model with $g(u)$, all H^1 solutions are in fact global in H^1 and uniformly bounded in H^1. This is a usual consequence of the conserved quantities and of the resolution of the Cauchy problem.

In these notes, we work in the previous framework, i.e. we consider global H^1 solutions that are limit of more regular global solutions. When stating an uniqueness result, we mean uniqueness in the class of the resolution of the Cauchy problem.

2.3. **One-soliton solution of the generalized models.** In this section, we describe the family of travelling waves or solitary waves of the generalized models introduced in Section 2.1. By abuse of language, we often refer to them as *solitons* though such terminology should be reserved to the integrable cases.

We begin with the gKdV equation, which is a one-dimensional problem. Looking for travelling wave solutions of the form $u(t,x) = Q_c(x - ct)$, one obtains the following equation for Q_c

$$Q_c'' + Q_c^p = cQ_c.$$

There exist localized solutions ($Q_c \in H^1(\mathbb{R})$) of this equation only for $c > 0$, as can be easily seen by Pohozaev's identity. In this case the solution is unique up to translation and is explicit:

$$Q_c(x) = c^{\frac{1}{p-1}} Q(\sqrt{c}x) \quad \text{where} \quad Q(x) = \left(\tfrac{p+1}{2}\cosh^{-2}\left(\tfrac{(p-1)x}{2}\right)\right)^{\frac{1}{p-1}}. \tag{2.15}$$

Thus, the only travelling wave solutions of (2.1) are $R_{c,x_0}(t,x) = Q_c(x - x_0 - ct)$.

As the gKdV equation, the P-BBM equation (2.12) has a two parameter family of solitary wave solutions: for any $c > 1$ and $x_0 \in \mathbb{R}$, $u(t,x) = \varphi_c(x - ct - x_0)$ is a travelling wave solution of (2.12) if φ_c is solution of

$$-c\partial_x^2\varphi_c + (c-1)\varphi_c - \varphi_c^2 = 0. \tag{2.16}$$

The unique even function going to zero at infinity which is solution of (2.16) is given by

$$\varphi_c(x) = (c-1)^{\frac{1}{p-1}} Q\left(\sqrt{\frac{c-1}{c}}x\right),$$

where Q is defined in (2.15) (for $p = 2$).

We now consider solitary waves of the nonlinear Schrödinger equation (2.6). There are of the form

$$u(t,x)=e^{\mathrm{i}\omega_0 t}Q_{\omega_0}(x), \tag{2.17}$$

for $\omega_0>0$, where $Q_{\omega_0}\in H^1(\mathbb{R}^d)$ is solution of

$$\Delta Q_{\omega_0}+Q_{\omega_0}^p=\omega_0 Q_{\omega_0},\quad Q_{\omega_0}>0. \tag{2.18}$$

In dimension $d=1$, the functions Q_{ω_0} are the same as for the KdV equation. Recall for any $d\geq 1$ that such positive solution of (2.18) exists and is unique up to translations (see [5], [22] and [30]), moreover, it is the solution of a variational problem. We call Q_{ω_0} the solution of (2.18) which is radially symmetric. By the symmetries of equation (2.6), for any $v_0\in\mathbb{R}^d$, $x_0\in\mathbb{R}^d$ and $\gamma_0\in\mathbb{R}$,

$$u(t,x)=Q_{\omega_0}(x-x_0-v_0t)e^{\mathrm{i}(\frac{1}{2}v_0.x-\frac{1}{4}|v_0|^2t+\omega_0t+\gamma_0)}$$

is also a solitary wave of (2.6), moving on the line $x=x_0+v_0t$. There may be other (non positive) solutions of (2.18) but we do not study the resulting solitary waves solutions here.

When considering the generalized models (i.e. (2.5) or (2.11)), we obtain the following general elliptic problem:

$$\Delta Q_\omega+f(Q_\omega^2)Q_\omega=\omega Q_\omega\quad\text{in }\mathbb{R}^d. \tag{2.19}$$

Let $F(r)=\int_0^r f(s)ds$. In the one-dimensional case $d=1$, a necessary and sufficient condition for existence of nontrivial solutions of (2.19) is known ([5]): there exists a solution of (2.19) in H^1 if and only if

$$r_0=\inf\{r>0,\text{ such that }F(r)=\omega r\}\text{ exists and satisfies }f(r_0)>\omega. \tag{2.20}$$

For $d\geq 2$ the elliptic problem is not as well understood as for $d=1$, see [5].

Comparing the family of solitons for the gKdV and the NLS equations, we remark that there are more parameters in the case of the NLS equation. In particular, we note that for $1<p<5$,

$$\int Q_c^2=c^{\frac{5-p}{2(p-1)}}\int Q^2, \tag{2.21}$$

and thus the speed of the soliton of the gKdV equation is directly related to its size in L^2. In contrast, the speed of a travelling wave and its size in L^2 can be chosen independently for the NLS equation, since they are related to independent parameters ω_0 and v_0, the velocity parameter v_0 being chosen by the Galilean invariance. This was clearly indicated in Zakharov and Shabat's paper [71].

2.4. Stability of one soliton.

In this section, we recall classical results of stability in H^1 of the one soliton solutions introduced in the previous section. By stability, we mean *orbital stability.* First, we state the result for the subcritical gKdV equation.

Theorem (Stability of the soliton for gKdV eq.). *Let $1<p<5$. Let $u(t)$ an H^1 solution of the gKdV equation* (2.1)*. For all $\epsilon>0$, there exists $\delta>0$, such that if $\|u(0)-Q\|_{H^1}\leq\delta$, then for all $t\in\mathbb{R}$, there exists $x(t)\in\mathbb{R}$, such that*

$$\|u(t)-Q(.-x(t))\|_{H^1}\leq\epsilon. \tag{2.22}$$

By invariance by scaling and translation of the gKdV equation, the result is the same for $Q_c(x-x_0)$. The proof of this result does not really use the equation but only the two H^1 conserved quantities, the L^2 norm and the energy (2.3)–(2.4), and the variational characterization of $Q(x)$. This stability result is due Benjamin [2], Bona [6], Cazenave and Lions [9] and Weinstein [68]). See below for a comparison of the two latter works.

For the NLS equations in the subcritical case, the result is completely similar.

Theorem (Stability of the soliton for the NLS eq.)**.** *Let*

$$1 < p < 1 + 4/d. \tag{2.23}$$

Let $u(t)$ be an H^1 solution of (2.6). Let $\omega_0 > 0$, $x_0 \in \mathbb{R}$, $v_0 \in \mathbb{R}^d$ and $\gamma_0 \in \mathbb{R}^d$. For all $\epsilon > 0$, there exists $\delta > 0$ such that if

$$\left\| u(0) - Q_{\omega_0}(. - x_0) e^{\mathrm{i}(\frac{1}{2} v_0 . x + \gamma_0)} \right\|_{H^1} \leq \delta$$

then, for all $t \in \mathbb{R}$, there exists $x(t), \gamma(t) \in \mathbb{R}$ such that

$$\left\| u(t,.) - Q_{\omega_0}(. - x(t)) e^{\mathrm{i}(\frac{1}{2} v_0 . x + \gamma(t))} \right\|_{H^1} \leq \epsilon.$$

By the invariances of the NLS equation, this property does not depend on v_0, x_0 nor on γ_0. Cazenave and Lions [9] proved a general stability result (implying the above theorem) for solitary waves which are minimizers, in a certain sense, of the energy functional, and when a compactness condition on minimizing sequences holds. This approach requires the concentration compactness method of P.L. Lions [34]. The condition obtained on f is sharp for the case of power nonlinearities $f(s^2) \equiv s^{p-1}$ (stability requires $1 < p < 5$).

By a different approach based on the expansion of the conservation laws around a solitary wave, Weinstein [68] also proved a general result of stability in H^1 in the case where Q_{ω_0} is a ground state under the non-degeneracy condition:

$$\frac{d}{d\omega} \int_{\mathbb{R}} Q_\omega^2(x)\, dx_{\big|\omega=\omega_0} > 0, \tag{2.24}$$

together with some assumptions on the spectrum of the linearized operator around Q_{ω_0}. These assumptions are checked in [68] for subcritical power nonlinearities for $d = 1$ and $d = 3$, and can also be checked under less restrictive conditions (see Maris [37] and McLeod [35]).

From [68], a natural assumption for nonlinear stability with a general nonlinearity is the existence of $\lambda > 0$ such that for any function $\eta \in H^1$:

$$\int \eta Q_\omega = \int \eta \nabla Q_\omega = 0 \Rightarrow \int \Big\{ |\nabla \eta|^2 + \omega |\eta|^2 - \big(f(Q_\omega^2) + 2Q_\omega^2 f'(Q_\omega^2)\big)|\eta|^2 \Big\} \geq \lambda \|\eta\|_{H^1}^2.$$

Note that in the pure power case this condition is equivalent to subcriticality.

Conversely, it is known from works of Grillakis, Shatah and Strauss [25] and Bona, Souganidis and Strauss [7] that if

$$\frac{d}{d\omega} \int_{\mathbb{R}} Q_\omega^2(x)\, dx_{\big|\omega=\omega_0} < 0,$$

then the solitary wave Q_{ω_0} is unstable in H^1, for the case of the gKdV equations, which corresponds to the condition $p > 5$. A proof of instability in the super critical case for the NLS equation is due to Berestyski and Cazenave [4]. The critical cases were treated by Weinstein [67] and Martel and Merle [40].

2.5. **Asymptotic stability results of Pego and Weinstein and Miller Weinstein.** By the previous results, we know that under a subcriticality assumption or equivalently condition (2.24), a soliton is a stable object in H^1. However, the stability result does not describe the exact behavior of the solutions that are in an H^1 neighborhood of the soliton. This description is related to the notion of asymptotic stability.

We introduce the following weighted space and norm

$$H^1_a = \{v \mid e^{ax}v \in H^1(\mathbb{R}),\ \text{with } \|v\|_{H^1_a} = \|e^{ax}v\|_{H^1}\}.$$

Pego and Weinstein [58] proved the following theorem for the gKdV equation.

Theorem (Asymptotic stability in H^1_a for the gKdV eq. [58])**.** *Let $c_0 > 0$, $x_0 \in \mathbb{R}$ and let $0 < a < \sqrt{c_0/3}$. There exists $\alpha_0 > 0$ such that if $u(t)$ is a global H^1 solution of (2.1) satisfying $u_0 \in H^2(\mathbb{R}) \cap H^1_a$,*

$$\|u(0) - Q_{c_0}(.-x_0)\|_{H^1} + \|u(0) - Q_{c_0}(.-x_0)\|_{H^1_a} \leq \alpha_0, \tag{2.25}$$

then, there exist $c_+ > 0$ with $|c_+ - c_0| \leq K_0\alpha_0$ and $x_+ \in \mathbb{R}$ with $|x_+ - x_0| < C\alpha_0$, such that for all $t \geq 0$, $v(t,x) = u(t,x) - Q_{c_+}(x - x_+ - c_+t)$ satisfies

$$\|v(t)\|_{H^1_a(x>\beta t)} \leq Ce^{-bt}, \tag{2.26}$$

for some constants $C,\ b > 0$.

We first comment on the location of the asymptotic soliton. The center of mass of the soliton contained in the solution $u(t)$ is asymptotically located at $x = x_+ + c_+t$, which means that it moves precisely on a line. We point out that this may not be generally the case for a solution in the energy space, see Theorem 3.2 below. It is here a consequence of the use of initial data in H^1_a.

Second, we note that if we assume $\|v(t)\|_{H^1(\mathbb{R})} \to 0$ as $t \to +\infty$, then $u(t)$ is exactly a soliton. Indeed, such a convergence in $H^1(\mathbb{R})$ would imply that $E(u) = E(Q_c)$ and $\|u\|_{L^2} = \|Q_c\|_{L^2}$ and then by the variational characterization of Q_c, $u(t)$ is a soliton of the form $Q_c(x - x_0 - ct)$. The lack of convergence in $H^1(\mathbb{R})$ is due to the presence of dispersion at the left $(x < 0)$ or to the possible presence of small solitons in the region $0 < x < ct$. In [58], the use of Sobolev spaces with exponential decay H^1_a avoids looking at the left of the soliton at the cost of a restrictive assumption on the initial data. For example, one cannot treat the initial data $u_0(x) = Q_c(x) + Q_{c'}(x - y_0)$, where $0 < c' < c$, c' small with respect to c, since such initial data has a weak exponential decay. Such initial data are of course of interest when studying multi soliton solutions.

In section 3, we present an asymptotic stability result where the use of exponential decay spaces is replaced by the local H^1 norm.

Miller and Weinstein [53] proved a similar result for the P-BBM equation (2.12). Note that (2.12) is not scaling invariant, and thus one cannot deduce the result for all the travelling waves from the proof for one soliton. In fact, their result is proved for *almost* all solitary waves, in a suitable sense.

2.6. **Similar results for the NLS equations.** A first result concerning the question asymptotic stability of one solitary wave for the nonlinear Schrödinger equation is due to Buslaev and Perelman [8], for $d = 1$ and under a set of conditions on the initial conditions, the solitary wave and the nonlinearity: the nonlinearity has to be flat at 0 ($|f(s^2)|s \leq s^q$, for $q \geq 9$) and the initial condition has to be close to a solitary wave in a weighted space (the norm is related to $\|u\|_{H^1}+\|xu\|_{L^2}+\|\hat{u}\|_{L^1}$). Perelman [60] addressed the same problem with two solitary waves, with large relative velocities. The other assumptions in these two works concern the spectrum of the linearized operator around the solitary waves and cannot be checked easily, in particular, these spectral assumptions are not simple consequences of (2.24) as in Weinstein's paper [68].

For $d\geq 3$, the question of asymptotic completeness of K solitary waves for nonlinear Schrödinger equations was considered simultaneously by Perelman [61] and Rodnianski, Schlag and Soffer [62], who prove similar results, both using dispersive estimates first due to Cuccagna. Both results require large velocities, flatness of $f(r)$ for r near 0, and some spectral assumptions (assumptions on the generalized null space of the linearized operator, nonexistence of nonzero eigenvalues and non resonance conditions). The closeness of the initial data to the sum of solitary waves is assumed in different norms. In [61], the initial data is in H^1 and close to the sum of solitary waves in the norm $\|u\|_{L^1}+\|\hat{u}\|_{L^m}$, for some $m>2$. In [62], closeness is required in the following norm: $\sum_{k=1}^{s}\|\nabla^k u\|_{L^1\cap L^2}$ for some $s>d/2$ integer.

As we can see, known asymptotic stability results for one or several solitary waves rely on spectral assumptions and on dispersive estimates in spaces strictly included in the energy space. Moreover, no result exists for $d=2$, or for any $d\geq 1$, with a pure power nonlinearity.

3. Asymptotic stability in the energy space for a soliton

As we have seen before, the stability result in H^1 does not describe the exact behavior of the solutions that are in an H^1 behavior of the soliton. Pego and Weinstein's result proves a local convergence to the soliton as time goes to $+\infty$ but require exponentially decaying initial data and was not proved for $p=4$. In this section, we recall the asymptotic stability result proved in [42], [46] and we discuss the same question for the P-BBM and NLS equations.

3.1. **Asymptotic stability for the gKdV equations.**

We first state the asymptotic stability result for the gKdV equation (2.1) with $p=2$, 3 or 4.

Theorem 3.1 (Asymptotic stability of one soliton for gKdV eq. [42], [46]). *Let $c_0>0$. There exists $K_0>0$ and for any $\beta>0$, there exists $\alpha_0=\alpha_0(\beta)>0$ such that the following is true. Let $u(t)$ be a global H^1 solution of (2.1) satisfying*

$$\|u(0)-Q_{c_0}\|_{H^1}\leq \alpha_0. \tag{3.1}$$

Then, there exist $c^+>0$ with $|c^+-c_0|\leq K_0\alpha_0$ and a C^1 function $x:[0,+\infty)\to\mathbb{R}$ such that

$$v(t,x)=u(t,x)-Q_{c^+}(x-x(t)) \quad \textit{satisfies} \quad \lim_{t\to+\infty}\|v(t)\|_{H^1(x>\beta t)}=0. \tag{3.2}$$

Moreover, $\lim_{t\to+\infty}\frac{dx}{dt}(t)=c^+$.

This result means that by taking α_0 small enough, we known the precise behavior of $u(t)$ on $x>\beta t$, for any $\beta>0$ (α_0 depending on β). Recall that strong convergence in $H^1(\mathbb{R})$ in (3.2) is not true since it would implies that $u(t)$ is a soliton (see the argument section 2.5). The region where the convergence in obtained in Theorem 1 is sharp since in the integrable case $p=2$ one can construct an explicit solution which behaves asymptotically as $t\to+\infty$ as $Q(x-t)+Q_c(x-ct)$, where $c>0$ arbitrary. In particular, choosing $c>0$ small, the H^1 norm of $Q_c(x-ct)$ is small, and this soliton travels on the line $x = ct$. This explains the necessity of a positive β in the convergence result. Moreover, one expects in general some loss of L^2 norm as dispersion for $x < 0$.

The virial type property or the linear rigidity property (see below) that is necessary to prove Theorem 1 has been checked only for equation (2.1) with p integer, in contrast with the stability result of Weinstein [68] which holds for general subcritical nonlinearities, and not only power nonlinearities.

Now, we focus on the behavior of $x(t)$ as $t \to +\infty$. One may think that the information on $x(t)$ in Theorem 1 is rather poor, since only $\frac{dx}{dt}(t)$ converges. Recall that in Pego and Weinstein's result (i.e. for exponentially decaying initial data), we have

$$x(t) - c^+ t - x^+ \to 0 \quad \text{as } t \to +\infty. \tag{3.3}$$

This convergence property is related to the decay at the right of the initial data and is not necessarily true in H^1. Indeed, it is not true in general for H^1 solutions that $|x(t) - c^+ t|$ converges as $t \to +\infty$ or even is bounded, by the following theorem.

Theorem 3.2 ([46]). *There exists $\alpha_0 > 0$ such that for any $0 < \alpha < \alpha_0$, the following is true. There exists an H^1 solution $u(t)$ of the KdV equation ((2.1) with $p = 2$) such that*

$$\sup_{t \in \mathbb{R}} \|u(t) - Q(x - x(t))\|_{H^1} \leq \alpha,$$

and

$$\lim_{t \to +\infty} \|u(t) - Q(x - x(t))\|_{H^1(x \geq t/2)} = 0,$$

for some C^1 function $x(t)$ satisfying, for some $\kappa > 0$,

$$\lim_{t \to +\infty} \frac{x(t) - t}{\sqrt{\log(t)}} = \kappa.$$

To prove Theorem 3.2, we use the special N–soliton solutions given explicitly in the case of the KdV equation, and for certain choice of parameters, we consider the limit case as $N \to +\infty$. The limit solution is composed for t negative of a soliton of size 1 plus a series of small solitons on the right of it. As time goes on, each small soliton interacts with the main soliton shifting it by a quantity related to the L^1 norm of the small solution. By a suitable choice of parameters, we obtain an infinite resulting shift, as in Theorem 3.2.

The rest of this section is devoted to the presentation of some tools used in the proof of Theorem 3.1.

3.2. **Decomposition of the solution and monotonicity.** We need two technical tools for this approach of the asymptotic stability. Consider a solution $u(t)$ of the gKdV equation such that

$$\alpha_0 \equiv \|u(0) - Q\|_{H^1} \tag{3.4}$$

is small enough. Recall first that from the H^1 stability result, there exists a constant $A_0 > 0$ such that (3.4) implies that for all $t \in \mathbb{R}$,

$$\|u(t) - Q(. - y(t))\|_{H^1} \leq A_0 \alpha_0, \tag{3.5}$$

for some function $y(t)$. A solution $u(t)$ satisfying (3.5) can be decomposed in the following sense.

Lemma 3.1 (Decomposition of a solution close to a soliton)**.** *If* $\alpha_0 > 0$ *is small enough then there exist* C^1 *functions* $c : \mathbb{R} \to (0, +\infty)$, $x : \mathbb{R} \to \mathbb{R}$, *such that*

$$\varepsilon(t,x) = u(t,x) - R(t,x), \quad \textit{where} \quad R(t,x) = Q_{c(t)}(x - x(t)), \tag{3.6}$$

satisfies, for all $t \in \mathbb{R}$,

$$\int R(t,x)\varepsilon(t,x)dx = \int (x - x(t))R(t,x)\varepsilon(t,x)dx = 0, \tag{3.7}$$

$$|c(t) - 1| + |c'(t)| + |x'(t) - c(t)| + \|\varepsilon(t)\|_{H^1} \le CA_0\alpha_0, \tag{3.8}$$

for some constant $C > 0$.

The function $v(t) = u(t) - Q(. - y(t))$ is small by (3.5). However, it is not true in general that $v(t) \to 0$ as $t \to +\infty$ in any sense, since the soliton in $u(t)$ could converge to some Q_{c^+} for c^+ close to 1, but not exactly 1. Remarkably, in Lemma 3.1 the orthogonality condition $\int \varepsilon Q = 0$ on ε selects a suitable scaling function $c(t)$ to prove the asymptotic stability; the rest of the proof consists in checking that $\varepsilon \to 0$, locally in H^1 and $c(t) \to c^+$.

Note that the equation of ε is easily obtained from the equations of Q and u:

$$\begin{aligned} &\partial_t\varepsilon + \partial_x^3\varepsilon + \partial_x\big((\varepsilon + R)^p - R^p\big) \\ &= -\frac{c'(t)}{2c(t)}\left(\frac{2R}{p-1} + (x - x(t))\partial_x R\right) + (x'(t) - c(t))\partial_x R. \end{aligned} \tag{3.9}$$

A second tool in the asymptotic stability result is a *monotonicity* result of local L^2 norm of a solution being such as in Lemma 3.1. Let $K > 0$. For $x \in \mathbb{R}$, let

$$\phi(x) = \frac{K}{\pi}\arctan(\exp(x/K)),$$

so that $\lim_{+\infty}\phi = 1$, $\lim_{-\infty}\phi = 0$ and for all $x \in \mathbb{R}$, $\phi(-x) = 1 - \phi(x)$. Note that by direct calculations

$$\phi'(x) = \frac{1}{K\pi\cosh(x/K)}, \quad \phi'''(x) \le \frac{1}{K^2}\phi'(x). \tag{3.10}$$

Let $0 < \sigma < 1/2$, $x_0 > 0$. We define, for $t_0 \in \mathbb{R}$, for all $t \le t_0$:

$$I_{x_0,t_0}(t) = \int u^2(t,x)\,\phi\left(x - x(t_0) + \sigma(t_0 - t) - x_0\right)dx,$$

We claim the following

Lemma 3.2 (Monotonicity result [42])**.** *For all* $0 < \sigma < \frac{1}{2}$, *for all* $K > \sqrt{\frac{2}{\sigma}}$, *if* $\alpha_0 > 0$ *is small enough, then for all* $t, t_0 \in \mathbb{R}$, $t \le t_0$,

$$I_{x_0,t_0}(t_0) - I_{x_0,t_0}(t) \le \theta\exp\left(-\frac{x_0}{K}\right), \tag{3.11}$$

for some constant θ.

The function $I_{x_0,t_0}(t)$ is thus almost decreasing. It plays the role of a Liapunov functional to study the convergence in regions $x > 0$ away from the soliton.

Proof. We repeat the proof from [42] and [46]. By simple calculations, for $f : \mathbb{R} \to \mathbb{R}$ of class C^3, we have (Kato's identity, see [27])

$$\frac{d}{dt}\int u^2 f = \int \Big(-3u_x^2 + \frac{2p}{p+1}u^{p+1}\Big)f' + \int u^2 f'''. \tag{3.12}$$

Let $0<\sigma<\frac{1}{2}$, $x_0>0$, $t_0\in\mathbb{R}$, and $K>\sqrt{\frac{2}{\sigma}}$. Let

$$\psi(t,x)=\phi(x-x(t_0)+\sigma(t_0-t)-x_0).$$

We obtain from (3.12), for all $t\leq t_0$,

$$\frac{d}{dt}\int u^2\psi=-\int\left(3u_x^2+\sigma u^2-\frac{2p}{p+1}u^{p+1}\right)\psi_x+\int u^2\psi_{xxx}.$$

Using (3.10) and $\frac{1}{K^2}\leq\frac{\sigma}{2}$, we obtain

$$\frac{d}{dt}\int u^2\psi\leq-\int\left(3u_x^2+\frac{\sigma}{2}u^2-\frac{2p}{p+1}|u|^{p+1}\right)\psi_x.$$

Let $R_0>0$ to be chosen later. For t,x such that $|x-x(t)|\geq R_0$, by $|Q(x)|\leq Ce^{-|x|}$, (3.8), and the well-known inequality $\|v\|_{L^\infty}^2\leq 2\|v_x\|_{L^2}\|v\|_{L^2}$, we have

$$|u(t,x)|\leq R(t,x)+\|u(t)-R(t)\|_{L^\infty}\leq Ce^{-\frac{R_0}{2}}+\sqrt{2}\alpha_3.$$

Therefore, for α_0 small enough and R_0 large enough, we have, for such t,x:

$$\frac{2p}{p+1}|u(t,x)|^{p-1}\leq\sigma/4.$$

Now, α_0 and R_0 are fixed to such values.

If $|x-x(t)|\leq R_0$ then $|x-x(t_0)+\sigma(t_0-t)-x_0|\geq-|x-x(t)|+|x(t)-x(t_0)+\sigma(t_0-t)-x_0|\geq-R_0+\frac{t_0-t}{2}+x_0$, and so

$$|\psi_x(t,x)|\leq Ce^{-\frac{t_0-t}{2K}}e^{-\frac{x_0}{K}}.$$

Therefore, by $\int|u|^{p+1}\leq C$, we obtain

$$\frac{d}{dt}\int u^2\psi\leq-\int\left(3u_x^2+\frac{\sigma}{4}u^2\right)\psi_x-Ce^{-\frac{t_0-t}{2K}}e^{-\frac{x_0}{K}}\leq-Ce^{-\frac{t_0-t}{2K}}e^{-\frac{x_0}{K}}. \tag{3.13}$$

By integration between t and t_0, we obtain (3.11).

One can see that the proof of Lemma 3.2 is rather elementary and uses only Kato's identity (3.12) and the exact form of the cut-off function ψ.

3.3. **Asymptotic stability as a consequence of the rigidity.**

In the more recent paper [46], the proof of the asymptotic stability is obtained from Lemmas 3.1, 3.2 and from a local virial type identity on the function ε. Roughly speaking a functional similar to $\int x\varepsilon^2(t,x)dx$ is used as a Liapunov functional around the soliton. We refer to [46] for this approach.

Here, we recall briefly the main steps of the approach used in [42] which is based on rigidity properties. We say that a solution $\widetilde{u}(t)$ of the gKdV equation is L^2 compact if

$$\forall\epsilon_0,\exists A_0>0,\forall t\in\mathbb{R},\quad\int_{|x|>A_0}\widetilde{u}^2(t,x+\widetilde{x}(t))dx\leq\epsilon_0, \tag{3.14}$$

for some $\widetilde{x}(t)$ with $\widetilde{x}'(t)>0$. By a remarkable property of the flow of the gKdV equation, condition (3.14) implies that $\widetilde{u}(t)$ is completely smooth (i.e. $\widetilde{u}(t)$ is C^∞ in time and space) and decays exponentially in x as well as all its derivatives.

Proposition 3.1 (L^2 compact solutions [42], [32])**.** *Let $p \geq 2$ integer. Let $\widetilde{u}(t)$ be a global H^1 solution of (2.1), globally bounded in H^1. Assume that there exist $0 < \beta_1 \leq \beta_2$ and a C^1 function $\widetilde{x}(t) : \mathbb{R} \to \mathbb{R}$ such that for all $t \in \mathbb{R}$, $\beta_1 \leq \widetilde{x}'(t) \leq \beta_2$, and such that (3.14) holds. Then $\widetilde{u} \in C^\infty(\mathbb{R} \times \mathbb{R})$. Moreover, there exist constants $\gamma > 0$ and $C_k > 0$ such that*

$$\forall k \in \mathbb{N}, \forall t \in \mathbb{R}, \forall x \in \mathbb{R}, \quad \left|\partial_x^k \widetilde{u}(t, x + \widetilde{x}(t))\right| \leq C_k \exp(-\gamma|x|). \tag{3.15}$$

This result is reminiscent of Kato smoothing effect, see [27]. Proposition 3.1 follows from techniques introduced in Martel and Merle [41] and [42] while proving the asymptotic stability result (see [32] for a complete proof).

We now state the rigidity result.

Theorem 3.3 (Liouville property close to Q [42])**.** *Let $p = 2$, 3 or 4. Let $\widetilde{u}(t)$ be a global H^1 solution of (2.1). There exists $\alpha_0 > 0$ such that if $\|\widetilde{u}(0) - Q\|_{H^1} \leq \alpha_0$ and if $\widetilde{u}(t)$ satisfies (3.14), then there exist $c_1 > 0$, $x_1 \in \mathbb{R}$ such that*

$$\widetilde{u}(t, x) = Q_{c_1}(x - x_1 - c_1 t).$$

From Proposition 3.1, assumption (3.14) is equivalent to an exponential decay assumption. It is a strong assumption on the solution. Theorem 3.3 states that if an H^1 solution of the gKdV equation is close in H^1 to a soliton, and has uniform exponential decay in x, then it is exactly a soliton.

The proof of the asymptotic stability result in H^1 in [42] follows from the rigidity property of the flow of the gKdV around the solitons, i.e. Theorem 3.3. We briefly explain this reduction. Let $u(t)$ be a solution such as in Theorem 3.1. By the uniform bound of $u(t)$ in H^1, one may consider a sequence $t_n \to +\infty$, a function $\widetilde{u}(0) \in H^1(\mathbb{R})$ and a positive real $\widetilde{c}_0 > 0$ such that

$$c^{-\frac{1}{p-1}}(t_n) u\Big(t_n, \tfrac{1}{\sqrt{c(t_n)}} . + x(t_n)\Big) \rightharpoonup \widetilde{u}(0) \quad \text{ and } \quad c(t_n) \to \widetilde{c}_0 \quad \text{ as } n \to +\infty.$$

Let $\widetilde{u}(t)$ be the solution of (2.1) defined for all $t \in \mathbb{R}$ corresponding to $\widetilde{u}(0)$. This solution is an asymptotic solution, and the fact that it comes asymptotically from the behavior of a solution $u(t)$ implies that $\widetilde{u}(t)$ is L^2 compact in the sense (3.14). The proof of this property uses Lemma 3.2 (see [42] for a detailed proof).

3.4. **Idea of the proof of the rigidity result.** By the previous discussion, to prove Theorem 3.1, we only have to prove Theorem 3.3. Under the assumptions of Theorem 3.3 one sees that $\widetilde{u}(t)$ can be decomposed with $\widetilde{\varepsilon}(t)$, $\widetilde{c}(t)$, $\widetilde{x}(t)$ and $\widetilde{R}(t)$ as $u(t)$ by Lemma 3.1. In particular $\widetilde{\varepsilon}$ satisfies the orthogonality conditions and the estimates of Lemma 3.1. Moreover, $\widetilde{\varepsilon}$, $\widetilde{c}(t)$, $\widetilde{x}(t)$ satisfy the same equation as ε, $c(t)$ and $x(t)$, i.e. equation (3.9). Finally, by assumption (3.14), Proposition 3.1 on $\widetilde{u}(t)$ and the exponential decay of Q, $\widetilde{\varepsilon}(t, x)$ also satisfies a uniform exponential decay property.

Note that the conclusion of Theorem 3.3 is equivalent to obtain $\widetilde{\varepsilon} \equiv 0$ provided that α_0 is small enough. We argue by contradiction on the smallness of α_0, assuming that there exists a sequence of solutions $(\widetilde{u}_n)$ which are not solitons, satisfying $\|\widetilde{u}_n(0) - Q\|_{H^1} \to 0$ as $n \to +\infty$ and the assumptions of Theorem 3.3. Let $\widetilde{\varepsilon}_n \not\equiv 0$ be associated to $\widetilde{u}_n$ by Lemma 3.1.

We assume in what follows that the following estimate on $\widetilde{\varepsilon}_n(t)$ holds (see [42]):

$$\sup_{t \in \mathbb{R}} \|\widetilde{\varepsilon}_n(t)\|_{H^1} \leq C \sup_{t \in \mathbb{R}} \|\widetilde{\varepsilon}_n(t)\|_{L^2}.$$

This allows us to prove that a renormalized version of the sequence $(\widetilde{\varepsilon}_n(t))$ converges to a solution $w(t) \not\equiv 0$ of the following linear problem

$$\partial_t w - \partial_x(\mathcal{L}w) = \alpha(t)\Big(\tfrac{2Q}{p-1} + xQ'\Big) + \beta(t)Q', \tag{3.16}$$

where $\alpha(t)$ and $\beta(t)$ are given function of t and

$$\mathcal{L}w = -\partial_x^2 w + w - pQ^{p-1}w,$$

is the linearized operator around Q. To obtain formally the equation of $w(t)$, it suffices to keep only the linear terms in the equation of $\widetilde{\varepsilon}(t)$.

Moreover, w also satisfies an exponential decay property

$$\forall y \in \mathbb{R}, \forall s \in \mathbb{R}, \quad |w(s,y)| \leq Ce^{-\theta|y|}, \tag{3.17}$$

and the same orthogonality conditions as $\widetilde{\varepsilon}_n(t)$.

To conclude the proof, it suffices to prove that necessarily $w \equiv 0$, which implies the desired contradiction. Therefore the last step of the proof of the asymptotic stability result is the classification of bounded and localized solutions of the linearized equation around Q.

Theorem 3.4 (Linear problem related to the gKdV eq. [42], [39])**.** *Let* $p > 1$. *Let* $w(t,x) \in C(\mathbb{R}, H^1(\mathbb{R})) \cap L^\infty(\mathbb{R}, H^1(\mathbb{R}))$ *be a solution of*

$$\partial_t w = \partial_x(\mathcal{L}w) + \alpha(t)\left(\tfrac{2}{p-1}Q + xQ'\right) + \beta(t)Q' \quad \text{on } \mathbb{R}\times\mathbb{R}, \tag{3.18}$$

where $\alpha(t)$ *and* $\beta(t)$ *are two continuous and bounded functions. Assume that for two constants* $C > 0$, $\sigma > 0$,

$$\text{for all } t, x \in \mathbb{R}, \quad |w(t,x)| \leq Ce^{-\sigma|x|}. \tag{3.19}$$

Then for all $t \in \mathbb{R}$,

$$w(t) \equiv a(t)\left(\tfrac{2}{p-1}Q + xQ'\right) + b(t)Q', \tag{3.20}$$

for some C^1 *bounded functions* $a(t)$ *and* $b(t)$ *satisfying*

$$a'(t) = \alpha(t), \quad b'(t) = -2\,a(t) + \beta(t). \tag{3.21}$$

Using this result and the orthogonality conditions on $w(t)$, we see that $a(t) = b(t)$ and thus $w \equiv 0$.

Theorem 3.4 has first been proved by Martel and Merle in [41] for $p = 5$ and [42] for $p = 2$, 3 and 4. The proof was technically complicated and required at some point to compute numerically the value of some integral. Moreover it was not clear how to generalize the proof to other p (e.g. $p > 5$ or p not integer). The author of the present paper has recently [39] given another proof that is general for any $p > 1$ and more elementary.

The proof presented in [39] is based on the fact that $w(t)$ such as in Theorem 3.4 is smooth, and $W(t) = \mathcal{L}w(t)$ satisfies

$$\partial_t W = \mathcal{L}(\partial_x W) - 2\alpha(t)Q.$$

It turns out that this equation is easier to handle than the equation of $w(t)$.

3.5. **P-BBM equation.** For the P-BBM equation (2.12), the same strategy as in [42] was used simultaneously by Mizumachi [56] and El Dika [18]-[19]. With some technical differences, but following the same line of proof, Mizumachi and El Dika proved that as for the gKdV equation, the asymptotic stability problem around φ_c is equivalent to a linear rigidity problem. In the case of the P-BBM equation, the linear problem to classify is

$$(1-\lambda\partial_x^2)\partial_t u = \partial_x\left(\mathcal{L}u\right) \quad \text{on } \mathbb{R}\times\mathbb{R}, \tag{3.22}$$

for $\lambda = \frac{c-1}{c} \in (0,1)$. In their respective works, Mizumachi and El Dika use a part of Miller and Weinstein's proof of asymptotic stability [53] to solve the linear problem. Unfortunately, by this argument, the proof cannot be finished for any value of $c>1$ (some isolated values may not work). Recently, the author of the present paper [39] could solve the linear rigidity problem without Miller and Weinstein's argument for equation (2.12), thus the result obtained is the following.

Theorem 3.5 (Asymptotic stability for the P-BBM eq. [56], [19], [39])**.** *Let* $c>1$. *Let* $u(t)$ *be an* H^1 *solution of* (2.12). *There exists* $\alpha_0>0$ *such that if*

$$\|u(0)-\varphi_c\|_{H^1}\le \alpha_0, \tag{3.23}$$

then, there exist c_+ *close to* c *and* $x(t)$ *such that*

$$u(t)-\varphi_{c_+}(\cdot - x(t)) \to 0 \quad \text{in } H^1(x>\tfrac{1+c_+}{2}t) \text{ as } t\to+\infty. \tag{3.24}$$

Note that the proof in [39] is only for equation (2.12) and not for the generalized forms of it. The asymptotic stability result is thus open for the generalized P-BBM equation, though it depends only on the linear property.

3.6. **Comments on the nonlinear Schrödinger equation.** We have already commented on existing asymptotic stability result for the NLS equations in Section 2.6. In particular, to our knowledge no such result exists for the nonlinear Schrödinger equation (2.6) with pure power nonlinearity. In fact, it is not clear whether or not such an asymptotic result should be true for the NLS equation. Indeed, if we consider the one dimensional cubic NLS equation (1.4), we known from [71] that explicit solutions containing parallel solitons exist. If one of the two parallel solitons can be taken arbitrarily small, it would prevent asymptotic stability to be true for the other soliton.

4. Existence of multi-soliton type solutions for generalized models

We have seen in the previous section that if the initial data $u(0)$ of a solution $u(t)$ of the subcritical gKdV equation is close to the function Q in H^1 norm then $u(t,.+x(t))$ converges in a local sense in H^1 to some close function Q_{c_+} for some center of mass $x(t)$. Moreover, if $u(t,.+x(t))$ converges to Q_{c_+} in $H^1(\mathbb{R})$ then $u(t)$ is exactly a soliton.

One can ask similar questions for a solution containing several solitons. Even the question of the stability in H^1 of several solitons was open until recently. Roughly speaking, when the solitons are decoupled (i.e. solitons with different speeds for large time) the results are similar to the ones for one soliton. The question of stability and asymptotic stability of the sum of several solitons is addressed in the next section. In this section, we concentrate on the construction of one solution that behaves exactly as the sum of several chosen solitons in $H^1(\mathbb{R})$.

4.1. **Multi-soliton for the gKdV.** The following result states existence and uniqueness of an asymptotically multi-soliton-like solution for the generalized KdV equation.

Theorem 4.1 (Asymptotic multi-solitons for the gKdV eq. [38])**.** *Let* $p = 2$, 3 *or* 4. *Let* $N \in \mathbb{N}$, $0 < c_1 < c_2 < \ldots < c_N$, *and* $x_1, \ldots, x_N \in \mathbb{R}$. *There exists one and only one function* $U \in C(\mathbb{R}, H^1(\mathbb{R}))$, *which is an* H^1 *solution of (2.1) in the sense of* [28] *and such that*

$$\lim_{t\to+\infty} \left\| U(t) - \sum_{j=1}^{N} Q_{c_j}(. - x_j - c_j t) \right\|_{H^1(\mathbb{R})} = 0. \tag{4.1}$$

Moreover, $U \in C(\mathbb{R}, H^s(\mathbb{R}))$ *for all* $s \geq 0$, *and there exist constants* $A_s > 0$ *such that for all* $s \geq 0$, *for all* $t \geq T_0$,

$$\left\| U(t) - \sum_{j=1}^{N} Q_{c_j}(. - x_j - c_j t) \right\|_{H^s(\mathbb{R})} \leq A_s e^{-\gamma t}, \tag{4.2}$$

where $\gamma = \sigma_0 \sqrt{\sigma_0}/32$ *and* $\sigma_0 = \min(c_1, c_2 - c_1, c_3 - c_2, \ldots, c_N - c_{N-1})$.

This result means that one can indeed construct a solution that behaves as $t \to +\infty$ as the sum of several solitons in $H^1(\mathbb{R})$. Moveover this solution is in fact completely smooth and converges exponentially in time in all Sobolev norms. Thus, the situation is quite similar to the one of the integrable cases $p = 2$ and $p = 3$, with the explicit multi-soliton solutions, *as far as asymptotic behavior is concerned.* However, we do not know the behavior of the solution $U(t)$ for $t \to -\infty$ or even backwards in time when the solitons begin to interact. This is the main difference with the integrable case.

Theorem 4.1 also contains a uniqueness result. Such a solution is unique in the class of solutions that behaves in the same way as $t \to +\infty$ in $H^1(\mathbb{R})$. This seems to be new even in the integrable case since results using the inverse scattering are for decaying and regular solutions.

Note that the proof of Theorem 4.1 does not use rigidity argument but only the stability of each soliton in the sense of Weinstein [69]. In particular, the result also holds for generalizations (2.5) of the model (2.1), under the only condition that independently the solitons are stable.

Mizumachi [55] has studied the case of two solitons with approximately the same sizes, his work is an evidence of a repulsive interaction between two solitons.

Very recently, Cote [11] studied the construction of a solution that behaves as the sum of N given solitons plus a linear part $T(t)v_0$ where v_0 is given.

For the P-BBM equation, and in fact also for the generalized version of it, El Dika and Martel [20] have proved exactly the same result as Theorem 4.1. The proof is quite similar, which is not surprizing since the two equations have a quite similar structure.

It is may be more surprizing that a similar general existence result holds for the NLS models. This is treated in the next section.

4.2. **Existence result for the NLS equations.** The following result holds for the NLS equation (2.6).

Theorem 4.2 (Existence of multi solitary waves for the subcritical NLSE [47])**.** *Let*

$$1 < p < 1 + 4/d. \tag{4.3}$$

Let $K \in \mathbb{N}^*$*. For any* $k \in \{1, \dots, K\}$*, let* $\omega_k^0 > 0$*,* $v_k \in \mathbb{R}^d$*,* $x_k^0 \in \mathbb{R}^d$ *and* $\gamma_k^0 \in \mathbb{R}$*. Assume that*

$$\text{for any } k \neq k',\ v_k \neq v_{k'}. \tag{4.4}$$

Let

$$R_k(t,x) = Q_{\omega_k^0}(. - x_k^0 - v_k t) e^{\mathrm{i}\left(\frac{1}{2} v_k . x - \frac{1}{4}|v_k|^2 t + \omega_k^0 t + \gamma_k^0\right)}. \tag{4.5}$$

Then, there exist an H^1 *solution* $U(t)$ *of* (2.6) *such that,*

$$\text{for all } t \geq 0, \quad \left\| U(t) - \sum_{k=1}^{K} R_k(t) \right\|_{H^1(\mathbb{R}^d)} \leq C e^{-\theta_0 t}, \tag{4.6}$$

for some $\theta_0 > 0$ *and* $C > 0$*.*

The existence result seems satisfactory since it holds for any dimension $d \geq 1$, and all subcritical p. Moreover, there is no condition on the parameters ω_k^0, v_k, γ_k^0 and x_k^0. Only (4.3), or more generally the stability of each soliton is required. In fact, we conjecture that this assumption could be removed from Theorem 4.2, but it is an open question so far.

There is no uniqueness result in Theorem (4.2). It is in fact an open problem whether or not such a solution is unique in H^1. This is certainly a difficult question related to the stability of several solitons in the next section.

4.3. Idea of the proof of existence.

Let $R_k(t)$ be K solitary waves of equation (2.6) as in (4.5) and let

$$R(t) = \sum_{k=1}^{K} R_k(t). \tag{4.7}$$

The construction of a solution $U(t)$ satisfying the conclusion of Theorem 4.2 is based on an asymptotic argument. Let $(T_n)_{n \geq 1}$ be an increasing sequence of $\mathbb{R}^+$ with $\lim_{n \to +\infty} T_n = +\infty$. For all $n \geq 1$, we consider u_n the unique global H^1 solution of

$$\begin{cases} \mathrm{i}\, \partial_t u_n = -\Delta u_n - |u_n|^{p-1} u_n, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u_n(T_n) = R(T_n). \end{cases} \tag{4.8}$$

The key point of the proof of Theorem 4.2 is the following uniform estimate on the sequence (u_n):

Proposition 4.1 (Uniform estimates)**.** *There exist* $T_0 > 0$*,* $C_0 > 0$*,* $\theta_0 > 0$ *such that, for all* $n \geq 1$*,*

$$\forall t \in [T_0, T_n], \quad \|u_n(t) - R(t)\|_{H^1} \leq C_0 e^{-\theta_0 t}. \tag{4.9}$$

Assuming this estimate, we prove easily that there exist $U_0 \in H^1(\mathbb{R}^d)$ and a subsequence $(u_{\phi(n)})$ of (u_n) such that

$$u_{\phi(n)}(T_0) \to U_0 \text{ in } L^2(\mathbb{R}^d) \text{ as } n \to +\infty. \tag{4.10}$$

We consider the global H^1 solution $U(t)$ of

$$\begin{cases} \mathrm{i}\,\partial_t U = -\Delta U - |U|^{p-1}U, & (t,x) \in \mathbb{R}\times\mathbb{R}^d, \\ U(T_0) = U_0. \end{cases} \tag{4.11}$$

Fix $t \geq T_0$. For n large enough, we have $T_n > t$ and by continuous dependence of the solution of (2.6) upon the initial data in $L^2(\mathbb{R}^d)$, we have

$$u_{\phi(n)}(t) \to U(t) \text{ in } L^2(\mathbb{R}^d) \text{ as } n \to +\infty \tag{4.12}$$

(Tsutsumi [64]). Thus, $u_{\phi(n)}(t) - R(t)$ converges weakly in $H^1(\mathbb{R})$ to $U(t) - R(t)$, and so by (4.9), we obtain

$$\|U(t) - R(t)\|_{H^1} \leq C_0 e^{-\theta_0 t}. \tag{4.13}$$

Thus, Theorem 5.2 is proved assuming Proposition 4.1.

For the proof of Proposition 4.1, we refer to the original paper ([47]). It is a large time stability statement, since $T_n \to +\infty$ and T_0 is fixed. However it is a weak stability statement since $u_n(T_n) = R(T_n)$ and since the interaction terms (between the various solitons) are exponentially small in time. The proof is based on a study of the variation in time of quantities (as energy and L^2 norm) related the Weinstein's proof of stability. To treat the case of various solitons, we have to consider localized versions of these quantities around each soliton.

5. Stability of the multi-soliton dynamics for large time

Once a multi-soliton solution has been constructed for large time, a natural question is to study its stability and asymptotic stability as was done in sections 2 and 3 for one soliton. For the gKdV equation and the P-BBM equation, the question has been answered in a satisfactory way. Some NLS equations with suitable nonlinearities and for $d = 1$, 2 and 3, could be addressed by similar but refined techniques, but the general question is open for pure power NLS equations.

The stability results stated in this section concern several solitons which are already ordered by increasing speeds and sufficiently decoupled (far enough the one from the others), moreover, we prove stability only for positive time. This means that we do not consider the case of solitons interacting.

5.1. Stability of multi-solitons for gKdV eq.

Theorem 5.1 (Asymptotic stability of the sum of N solitons [48]). *Let $p = 2$, 3 or 4. Let $0 < c_1^0 < \ldots < c_N^0$. There exist $\gamma_0, A_0, L_0, \alpha_0 > 0$ such that the following is true. Let $u(t)$ be an H^1 solution of* (2.1), *and assume that there exist $L > L_0$, $0 < \alpha < \alpha_0$, and $x_1^0 < \ldots < x_N^0$, such that*

$$\left\| u(0) - \sum_{j=1}^{N} Q_{c_j^0}(. - x_j^0) \right\|_{H^1} \leq \alpha, \quad \textit{and } x_j^0 > x_{j-1}^0 + L, \textit{ for all } j = 2, \ldots, N. \tag{5.1}$$

Then, there exist $x_1(t), \ldots, x_N(t)$ such that

(i) Stability of the sum of N decoupled solitons.

$$\forall t \geq 0, \quad \left\| u(t) - \sum_{j=1}^{N} Q_{c_j^0}(x - x_j(t)) \right\|_{H^1} \leq A_0 \left(\alpha + e^{-\gamma_0 L} \right). \tag{5.2}$$

(ii) Asymptotic stability of the sum of N solitons. There exist $c_1^{+\infty},\dots,c_N^{+\infty}$, with $|c_j^{+\infty}-c_j^0|\le A_0\left(\alpha+e^{-\gamma_0 L}\right)$, such that

$$\left\|u(t)-\sum_{j=1}^{N}Q_{c_j^{+\infty}}(x-x_j(t))\right\|_{H^1(x>c_1^0t/10)}\to 0,\quad \dot{x}_j(t)\to c_j^{+\infty}\quad \text{as } t\to+\infty. \tag{5.3}$$

As in Theorem 3.1, one cannot expect the convergence to hold in $H^1(x>0)$. Indeed, assumption (5.1) on the initial data allows the existence in $u(t)$ of an additional soliton of size less that α (thus travelling at arbitrarily small speed). For $p=2$, an explicit example can be constructed using the N–soliton solutions. Moreover, convergence in $H^1(\mathbb{R})$ would imply that $u(t)$ is a special solution $U(t)$ as constructed in Theorem 4.2.

As a direct corollary of Theorem 1, for $p=2$ and $p=3$, the explicit multi-soliton solutions are stable and asymptotically stable, see Corollary 1 in [42]. This improves the result in [36].

5.2. **Stability of multi-solitons for the P-BBM equation.** For the P-BBM equation (2.12), the stability and asymptotic stability result of multi-solitons is exactly the same as for the gKdV equations, with a very similar proof, see El Dika and Martel [20], and [39] for the asymptotic stability result without restriction on the speeds.

5.3. **Stability of multi-solitons for Nonlinear Schrödinger equations.** In this section, we study similar questions for the nonlinear Schrödinger equation. The situation seems to be more delicate in general in this case. We have already observed (section 3.6) that no asymptotic stability result in the energy space for the nonlinear Schrödinger was known. In the case of multi-soliton dynamics, it seems that the richness of the family of solitons is an extra difficulty with respect to the gKdV equation. Indeed, we recall that for the nonlinear Schrödinger equation with pure power nonlinearity, the speed of the soliton is not related to its scaling, thus it might possible for one or several small solitons to accompany for long time a large soliton. Such behavior is not possible for the gKdV equation (nor for the P-BBM equation) since the speed of the soliton is related to its scaling, which implies in a short time a spatial decoupling between large and small objects.

The results obtained so far concerning the stability in the energy space of multi-soliton for the nonlinear Schrödinger equation get around this difficulty by imposing conditions on the nonlinearity so that small solitons do not perturb the general dynamics of large solitons.

Case of dimension 1.

We first state the main result in the one dimensional case.

Theorem 5.2 (Stability of the sum of K solitary waves of NLS in 1D [49])**.** *Assume that $f:\mathbb{R}\to\mathbb{R}$ is a function of class C^1 such that $f(0)=0$ and*

***(A1) Flatness at** 0:*

there exists $C>0$ such that for all $r\in[0,1]$, $f'(r)\le Cr$.

Let $K\in\mathbb{N}$ and for all $k\in\{1,\dots,K\}$, let $\omega_k^0>0$ be such that there exists $Q_{\omega_k^0}\in H^1(\mathbb{R})$ a positive solution of

$$Q_\omega''+Q_\omega^p=\omega_k^0Q_\omega,\quad Q_\omega>0. \tag{5.4}$$

satisfying

(A2) Nonlinear stability of each wave:

$$\frac{d}{d\omega}\int_{\mathbb{R}} Q_\omega^2(x)\,dx_{|\omega=\omega_k^0} > 0.$$

For all $k\in\{1,\ldots,K\}$, *let* $x_k^0\in\mathbb{R}$, $\gamma_k^0\in\mathbb{R}$ *and* $v_k\in\mathbb{R}$, *with* $v_1<v_2<\ldots<v_K$. *Assume further that, for all* $k\in\{1,\ldots,K-1\}$,

(A3) Condition on relative speeds: $(v_{k+1}-v_k)^2 > 4\,|\omega_{k+1}^0-\omega_k^0|$.

Let $u(t)$ *be an* H^1 *solution of* (2.6). *There exist* $L_0>0$, $\theta_0>0$, $A_0>0$, *and* $\alpha_0>0$ *such that for any* $L>L_0$, *and* $0<\alpha<\alpha_0$, *if*

$$\left\|u(0)-\sum_{k=1}^K Q_{\omega_k^0}(.-x_k^0)e^{\mathrm{i}(\frac12 v_k x+\gamma_k^0)}\right\|_{H^1}\leq\alpha, \tag{5.5}$$

and if for all $k\in\{1,\ldots,K-1\}$,

$$x_{k+1}^0-x_k^0>L, \tag{5.6}$$

then $u(t)$ *is defined in* H^1 *for all* $t\geq 0$ *and there exist functions* $x_1(t),\ldots,x_K(t)\in\mathbb{R}$, *and* $\gamma_1(t),\ldots,\gamma_K(t)\in\mathbb{R}$, *such that for all* $t\geq0$,

$$\left\|u(t)-\sum_{k=1}^K Q_{\omega_k^0}(.-x_k(t))e^{\mathrm{i}\left(\frac12 v_k x+\gamma_k(t)\right)}\right\|_{H^1}\leq A_0\big(\alpha+e^{-\theta_0 L}\big). \tag{5.7}$$

Moreover, for all $t\geq0$,

$$|\dot x_k(t)-v_k|+\left|\dot\gamma_k(t)-\Big(\omega_k^0-\frac{v_k^2}{4}\Big)\right|\leq A_0\big(\alpha+e^{-\theta_0 L}\big). \tag{5.8}$$

Our first comment is that, in contrast with previously existing results for the nonlinear Schrödinger equation, Theorem 5.2 is a stability result *in the energy space.* Moreover, no spectral assumption on the linearized operator is required, except the natural assumption that the various solitary waves are independently nonlinearly stable.

We now comment on assumption (A1) concerning the nonlinearity. The assumption on f which is really used in the proof of Theorem 5.2 is

$$\text{There exists } C>0 \text{ such that, for all } s\in[0,1],\quad f(s^2)s^2-F(s^2)\leq Cs^6,$$

which is a consequence of (A1).

Recall that in the case of a pure power nonlinearity $f(s^2)\equiv s^{p-1}$, the critical exponent for the stability of the solitary waves is $p=5$, which means that the stability condition (A2) holds on the solitary waves if and only if $1<p<5$. But for $f(s^2)\equiv s^{p-1}$, assumption (A1) requires $p\geq5$. This means that Theorem 5.2 does not apply to the pure power case, for any p.

However, as pointed out in [49], there are important explicit examples of nonlinearities f to which Theorem 5.2 applies. Let us recall here one such class of examples constructed from the pure power case. Let $1<p<5$ and $q\geq5$. Consider $f(s^2)=s^{p-1}$, for $s>s_0$, $f(s^2)=s^{q-1}$, for $0\leq s<\frac{s_0}{2}$ and f increasing and of class C^1. For $s_0>0$ small, equation (2.6) is a perturbation of the pure power subcritical Schrödinger equation and (A1) holds since $q\geq5$. Thus, since (A2) is true for any solitary wave for $f(s^2)=s^{p-1}$ and since such condition depends continuously on f, it follows that for small s_0, the solitary waves such that $\omega>\omega_0>0$

are stable in the sense that (A2) is true. Therefore provided that assumption (A3) on the speeds is satisfied, Theorem 5.2 applies to this case.

Assumption (A3) means that if ω_k^0 and ω_{k+1}^0 are different then the relative speed of the corresponding solitary waves, i.e.: $v_{k+1} - v_k$ has to be sufficiently large. However any speeds $v_k < v_{k+1}$ are possible if $\omega_k^0 = \omega_{k+1}^0$. Note that condition (A3) is invariant by the Galilean transform (2.10).

Case of dimensions 2 and 3.

Now, we turn to the nonlinear Schrödinger equation (2.11) set in $\mathbb{R}^d$ for $d = 2$ or 3. From Weinstein's stability proof ([68]), a natural assumption for nonlinear stability is the existence of $\lambda > 0$ such that for any real-valued function $\eta \in H^1$:
(A2')

$$\int \eta Q_\omega = \int \eta \nabla Q_\omega = 0 \Rightarrow \int \Big\{ |\nabla \eta|^2 + \omega|\eta|^2 - \big(f(Q_\omega^2) + 2Q_\omega^2 f'(Q_\omega^2)\big)|\eta|^2 \Big\} \geq \lambda \|\eta\|_{H^1}^2.$$

Note that this condition is equivalent to subcriticality in the pure power case.

We claim the following result.

Theorem 5.3 (Stability of the sum of K solitary waves in 2D and 3D [49]). *Let $d = 2$ or 3. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a function of class C^1 such that for some constant $C > 0$:*

(A1') *for all* $r \geq 0, \quad f'(r) \leq Cr.$

Let $K \in \mathbb{N}$ and for all $k \in \{1, \dots, K\}$, let $\omega_k^0 > 0$ be such that $Q_{\omega_k^0} > 0$ is a solution of

$$\Delta Q_{\omega_0} + Q_{\omega_0}^p = \omega_0 Q_{\omega_0}, \quad Q_{\omega_0} > 0. \tag{5.9}$$

*satisfying **(A2')**. For all $k \in \{1, \dots, K\}$, let $x_k^0 \in \mathbb{R}^d$, $\gamma_k \in \mathbb{R}$, and $v_k \in \mathbb{R}^d$ satisfying*

$$\text{for all } k \neq k', \quad v_k \neq v_{k'}. \tag{5.10}$$

Let $u(t)$ be an H^1 solution of (2.6). There exists $\omega_0 = \omega_0(v_1, \dots, v_K) > 0$, $T_0 > 0$, $\theta_0 > 0$, $A_0 > 0$, and $\alpha_0 > 0$ such that if

(A3') *for all* $k \neq k', \quad |\omega_k^0 - \omega_{k'}^0| < \omega_0,$

and, for any $T > T_0$, and $0 < \alpha < \alpha_0$, if

$$\left\| u(0) - \sum_{k=1}^K Q_{\omega_k^0}(. - x_k^0 - v_k T) e^{\mathrm{i}(\frac{1}{2} v_k . x + \gamma_k^0)} \right\|_{H^1} \leq \alpha, \tag{5.11}$$

then, $u(t)$ is defined in H^1 for all $t \geq 0$ and there exist $x_1(t), \dots, x_K(t) \in \mathbb{R}^d$, and $\gamma_1(t), \dots, \gamma_K(t) \in \mathbb{R}$, such that, for all $t \geq T$,

$$\left\| u(t) - \sum_{k=1}^K Q_{\omega_k^0}(. - x_k(t)) e^{\mathrm{i}(\frac{1}{2} v_k . x + \gamma_k(t))} \right\|_{H^1} \leq A_0 \big(\alpha + e^{-\theta_0 T}\big). \tag{5.12}$$

Moreover, for all $t \geq T$,

$$|\dot{x}_k(t) - v_k| + \left| \dot{\gamma}_k(t) - \Big(\omega_k^0 - \frac{|v_k|^2}{4}\Big) \right| \leq A_0 \big(\alpha + e^{-\theta_0 T}\big). \tag{5.13}$$

This result is the analogue of Theorem 5.2, except that the condition on the relative speeds (A3') is less explicit.

To our knowledge, Theorem 5.3 is the first result of stability of the sum of solitary waves for semilinear Schrödinger equations in space dimension 2. Moreover, as Theorem 5.2, Theorem 5.3 holds in the energy space and without spectral assumptions. Therefore, for $d = 3$, it is different from the existing results described in section 3.6 ([61] and [62]).

We have explained in section 4.3 how localization arguments allowed to prove the uniform estimates in the proof of existence in Theorem 4.2. Recall that in the context of section 4.3, the rest has exponential decay in time. The proof of the stability results Theorems 5.2–5.3 is also based on localization arguments but they are more delicate since the rest is of size α. This is where we need the flatness condition on f near 0.

6. Comments

As we mentioned in the Introduction, for the gKdV equation (and for the similar model due to Peregrine, and Benjamin, Bona and Mahony), the behavior of the solutions *close to the sum of solitons* and *asymptotically as* $t \to +\infty$ is well-known. However, no analysis for other situations exist in the non integrable cases. In particular, whether or not any solution of the gKdV equation should decompose as $t \to +\infty$ as the sum of solitons in the right ($x > 0$) is an open question in the non integrable case, except for solutions that are initially close to the sum of several solitons.

Another vast question when the model is not integrable is the behavior of the asymptotic multi-soliton-like solutions backwards in time, i.e. when the solitons begin to interact at a finite distance. Note that this question was already mentioned in Miura's review [54].

Finally, there are recent results for the nonlinear Schrödinger equations ([61], [62] and [49], [47]), yet most of the questions concerning the behavior of solutions close to the sum of solitons are open. In particular, the flow around solitons is not understood for general NLS equations.

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Université de Versailles-Saint-Quentin-en-Yvelines, 45, avenue des Etats-Unis, 78035 Versailles - France

E-mail address: `martel@math.uvsq.fr`